Connectedness in Metric Space

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Connectedness is a powerful tool in proofs of well-known results. Roughly speaking, a connected metric space (or, a connected subspace of a metric space) is one that is a "single piece". This is a very difficult notion to be formulated precisely. If we look at $\mathbb{R}\backslash\{0\}$, we would think of it consisting of two pieces, namely, one of positive numbers and the other of negative numbers. Similarly, if we consider an ellipse or a parabola, it is in a single piece while a hyperbola has two distinct pieces. If we remove a single point from a circle, it still remains as a single piece. An attempt to generalize the above property leads to the concept of connected metric spaces.

There are many different concepts of connectedness; each one is important in some area of study. We discuss connectedness, path connectedness, polygonally connectedness and locally connectedness in the lecture.

Definition 1.

Let (X, d) be a metric space. X is said to be disconnected if there are non-empty, disjoint, open sets U, V of X such that $U \cup V = X$. That is, a metric space which is a union of two disjoint non-empty open sets is called disconnected.

The metric space X is said to be connected if it is not disconnected.

Theorem 2 (Characterizations of Connected Metric Spaces).

Let (X, d) be a metric space. Then the following are equivalent :

1. *X is disconnected ;*
2. *There exist non-empty, disjoint, open sets U, V of X such that $U \cup V = X$;*
3. *There exist non-empty, disjoint, closed sets U, V of X such that $U \cup V = X$;*
4. *There exists a non-empty proper subset of X which is both open and closed ;*
5. *(An important characterization of connected spaces.) Every continuous function $f : X \to \{0, 1\}$ is constant (where $\{0, 1\}$ has the discrete topology).*
6. *There exist two disjoint proper non-empty subsets U and V such that U and V are both open and closed in X and $X = U \cup V$. In such a case, we say that the pair (U, V) is a disconnection of X.*
7. *There exist disjoint non-empty subsets U and V of X such that $X = U \cup V$ and $\overline{U} \cap V = U \cap \overline{V} = \emptyset$.*

Connected Subsets of Metric Spaces

To understand what it means for a *subset* of a metric space to be connected, we note that we can make a subset of a metric space into a metric space in its own right, using the same measure of distance on the subset that is used on the full metric space. That is, given a metric space (X, d) and a subset $A \subseteq X$, we get the metric space (M, d'), where $d'(x, y) = d(x, y)$ for x and y in A. When we do this, we will refer to A as a subspace of X rather than simply a subset. (In practice, we would usually just write d and not d', we will use d' to make it clear whether we are talking about the original metric space or a subspace.)

Theorem 3.

Let (X, d) be a metric space and A be a subset of X. A subset U of A is open in the subspace (A, d') if and only if there is an open subset $\mathcal{O}$ in X such that $U = A \cap \mathcal{O}$. A subset F of A is closed in the subspace (A, d') if and only if there is a closed subset $\mathcal{C}$ in X such that $F = A \cap \mathcal{C}$.

Connected Subsets

A subtlety is that if A is a subspace of a metric space X, then open means with respect to A, not with respect to X. For example, consider $A = \{0, 1\} \subseteq \mathbb{R}$. Then A is disconnected: take $U = \{0\}$ and $V = \{1\}$. Here U and V are open in A even though they are not open in $\mathbb{R}$. The following result says that if A is a subset of a metric space X, then one can, in fact, also check connectedness by working with sets that are open in X.

Thus connectedness is a property that belongs properly to a metric space ; it is not relative to any metric superspace that the space may sit inside.

Theorem 4.

Let (X, d) be a metric space and let $A \subseteq X$. Then A is disconnected in the metric space (A, d') iff there are disjoint sets U, V of X that are open in (X, d) and such that $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$ and $A \subseteq U \cup V$.

Theorem 5.

Let Z be a metric subspace of a metric space X and $S \subseteq Z$. Then S is a connected subset of X iff S is a connected subset of Z.

If $A = \{0, 1\} \subseteq \mathbb{R}$, then A is disconnected since we can take $U = (-1/2, 1/2)$ and $V = (1/2, 3/2)$.

Theorem 6.

A subset A of a metric space X, endowed with the subspace topology, is connected iff every continuous function $f : A \to \{0, 1\}$ is a constant function (where $\{0, 1\}$ has the discrete topology called discrete two point space).

Connected Subsets

- Let X be a set such that $|X| \geq 2$ with discrete metric. Then X is not connected.
- Let A be a finite subset of a metric space X with $|A| \geq 2$. Then A is not connected.
- Let A be any countable subset of a metric space X with $|A| \geq 2$. Then A is not connected. Indeed, let $a, b \in A$ with $a \neq b$. Since the interval $(0, d(a, b))$ is uncountable, there exists $s \in (0, d(a, b))$ such that no point of X is of distance s from a. Then $B(a, s) = B[a, s]$, so it is both open and closed in X. But $a \in B(a, s)$ and $b \notin B(a, s)$. Note that $B(a, s) = \{x \in X : d(x, a) < s\}$ and $B[a, s] = \{x \in X : d(x, a) \leq s\}$.

Connected Subsets of $\mathbb{R}$

On the real line, connected sets must be of a certain form. Recall that a set $X \subseteq \mathbb{R}$ is an *interval* iff for any $a, b \in X$ with $a \leq b$, if $a < x < b$, then $x \in X$.

The term "interval" includes bounded intervals of the form $[a, b], (a, b), [a, b)$, or $(a, b]$, as well as infinite intervals of the form $(-\infty, a], (-\infty, a), (a, \infty), [a, \infty)$ or $(-\infty, \infty)$. The empty set is also considered to be an interval.

Any interval is a connected subset of $\mathbb{R}$. In fact, intervals are the only connected subsets of $\mathbb{R}$ with the usual topology.

Theorem 7.

If $X \subseteq \mathbb{R}$, then X is connected iff it is an interval.

Connected Sets

If we remove a point from the interior of an interval in $\mathbb{R}$, we get a disconnected set: although $[0,3]$ is connected, $[0,1) \cup (1,3]$ is disconnected. This is not true in higher dimensions. For example, $D = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$, which is the unit disk in $\mathbb{R}^2$, is connected, but also so is $D \backslash \{(0,0)\}$, which is the unit disk with the origin removed. This is an important topological difference between $\mathbb{R}$ and Euclidean spaces of higher dimension.

Continuous Image of Connected Sets

It is easy to show that connectedness, like compactness, is preserved by continuous functions. That is, the continuous image of a connected metric space is connected.

Theorem 8.

Let (A, ρ) and (B, τ) be metric spaces, and suppose that $f : A \to B$ be a continuous function from A to B. If A is connected, then its image $f(A)$ is also connected.

Proof. Let $f : g(X) \to \{\pm 1\}$ be a continuous function on $f(X)$. Since $f \circ g : X \to \{\pm 1\}$ is continuous and X is connected, it follows that $f \circ g$ is a constant on X. Hence f is a constant on $g(X)$. Therefore $g(X)$ is connected.

Aliter : Assume that $g(X)$ is not connected. Then there exists nonempty proper subset $V \subseteq g(X)$ which is both open and closed in $g(X)$. Since f is continuous, $g^{-1}(V)$ and $g^{-1}(g(X) \backslash V)$ are both nonempty, closed and open in X. This contradicts the hypothesis that X is connected.

Continuous Image of Connected Sets

As a consequence of the previous result, we see that connectedness is a topological property. Connectedness is preserved by metric homeomorphism.

We may use this fact to distinguish between some non-homeomorphic spaces. For example, the space $[0, 1]$ and $(0, 1)$ (both with the subspace topology as subsets of $\mathbb{R}$) are not homeomorphic because removing any point from $(0, 1)$ gives a disconnected space, whereas removing an end-point from $[0, 1]$ still leaves an interval which is connected.

Theorem 9.

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Assume that y is a point between $f(a)$ and $f(b)$, that is, either $f(a) \leq y \leq f(b)$ or $f(b) \leq y \leq f(a)$ holds. Then there exists $x \in [a, b]$ such that $f(x) = y$.

The following converse of the Intermediate Value Theorem also holds.

Theorem 10.

Let X be a metric space. If every continuous function $f : X \to \mathbb{R}$ has the intermediate value property (i.e., if $y_1, y_2 \in f(X)$ and y is a real number between y_1 and y_2, then there exists an $x \in X$ such that $f(x) = y$), then X is a connected metric space.

Applications of Intermediate Value Theorem

Theorem 11 (Fixed Point Theorem).

Let $f : [0,1] \to [0,1]$ be a continuous function. Then there exists a point $c \in [0,1]$ such that $f(c) = c$.

Theorem 12 (Existence of n-th roots).

Let $\alpha \in [0,\infty)$ and $n \in \mathbb{N}$. Then there exists a unique $x \in [0,\infty)$ such that $x^n = \alpha$.

Theorem 13.

Any polynomial with real coefficients and of odd degree has a real root. That is, if $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $a_j \in \mathbb{R}$ for $0 \leq j \leq n$, $a_n \neq 0$ and n is odd, then there exists $\alpha \in \mathbb{R}$ such that $p(\alpha) = 0$.

Connected Sets

It is certainly not true that the union of connected sets is connected. (Just consider $[1, 2] \cup [3, 4]$.) However, if a collection of connected sets have a non-empty intersection, then the union is connected. That is, fastening together connected spaces "with an overlap" gives a connected space.

Theorem 14.

Let $\{A_i : i \in I\}$ be a collection of connected subsets of a metric space X with the property that for all $i, j \in I$ we have $A_i \cap A_j \neq \emptyset$. Then $A := U_i A_i$ is connected. Deduce that if $\{A_i : i \in I\}$ is a collection of connected subsets of X such that $\bigcap_{i \in I} A_i \neq \varnothing$, then the union $\bigcup_{i \in I} A_i$ is connected.

Theorem 15.

Let f be a real-valued continuous function defined on a metric space X. At each point of a subset A of X, $f(x)$ is either equal to $+1$ or to -1. Then A is connected iff every such f is constant on A.

Theorem 16.

Let X_i $(i = 1, 2, \ldots, n)$ be connected non-empty metric spaces. Endow the product $P = \prod_{i=1}^{n} X_i$ with a product metric. Then P is connected iff each X_i is connected, $i = 1, 2, \ldots, n$.

Theorem 17.

Let A be a connected subset of a metric space X. Let $A \subseteq B \subseteq \overline{A}$. Then B is connected.

Connected Components

One can start with a singleton set or any connected subset A of a metric space X. Can we have a maximal connected subset M of X containing A? Maximal in the sense that M cannot be a proper subset of any connected subset of X. By Zorn's lemma, such a maximal connected subset exists, we call it as *connected component*, or simply a component.

Connected components partitiion the space in that they are mutually disjoint and their union is the whole space ; they are always closed, but need not be open.

How to find a connected component containing an element x ?

Let $x \in X$. The union $C(x)$ of all connected subsets of X containing x is a maximal connected subset of X.

Connected Components

- A metric space X is connected iff only connected component is X.
- In a discrete metric space, every singleton set is both open and closed and so has no proper superset that is connected. Therefore discrete metric spaces have the property that their connected components are their singleton subsets.
- In an arbitrary metric spcae, there may be any number of singleton connected components, but every other connected component (other than singleton sets) must be uncountable.
- Every element of a metric X lies in a unique connected component and X is the disjoint union of connected components.
- A metric space whose only connected subsets are singleton sets is called *totally disconnected*.

Connected Components

Theorem 18.

Let X be a metric space. Then

1. *each connected subset of X is contained in exactly one connected component ;*
2. *each nonempty connected subset of X that is both open and closed in X is a connected component of X ;*
3. *each connected component of X is closed ;*
4. *the connected components of X are mutually disjoint ;*
5. *X is the union of its connected components.*

We have seen that connected components are closed. Can they be open ? They are open if there is only a finite number of connected components, but no in general. Consider $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$, a subset of $\mathbb{R}$. $\{0\}$ is a connected component of X which is closed in X, but it is open because every open interval containing 0 will contain $\frac{1}{n}$ for some $n \in \mathbb{N}$.

Connected Components

1. The components of the space $[0, 1] \cup [2, 3]$ with the subspace topology inherited from $\mathbb{R}$, are the subspaces $[0, 1]$ and $[2, 3]$.
2. Components of $\mathbb{Z}$ (with the subspace topology for $\mathbb{R}$) are the singleton sets. Hence $\mathbb{Z}$ is totally disconnected.
3. Components of $\mathbb{Q}$ (with the subspace topology for $\mathbb{R}$) are the singleton sets. Hence $\mathbb{Q}$ is totally disconnected.
4. The components of X with the discrete metric are singleton sets. Hence it is totally disconnected.
5. The Cantor set K is totally disconnected : any two elements a, b of K with $a < b$, there is some $x \in \mathbb{R} \backslash K$, so that a and b do not belong to the same connected component of K.

Proposition 19.

$S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ *is connected.*

Proof : The strategy is to show that S^n is the union of the closed upper and lower hemispheres, each of which is homeomorphic to the closed unit disk in $\mathbb{R}^n$ and to observe that the hemispheres intersect. Let $S^n_+ := \{x \in \mathbb{R}^{n+1} : x_{n+1} \geq 0\}$ be the upper hemisphere. Let $D_n := \{u \in \mathbb{R}^n : \|u\| \leq 1\}$. Note that D_n is convex and hence connected. We claim that D_n is homeomorphic to S^n_+. Consider the map $f_+ : D_n \to S^n_+$ given by $f_+(u) = (u_1, \dots, u_n, \sqrt{1 - \|u\|^2})$. Clearly, f_+ is bijective and continuous. Since f_+ is a bijective continuous map of a compact space to a metric space, it is a homeomorphism. In any case, S^n_+ being the continuous image of the connected set D_n, is connected. Similarly, we show that the lower hemisphere $S^n_- := \{x \in \mathbb{R}^{n+1} : x_{n+1} \leq 0\}$ is the image of $f_- : D_n \to S^n_-$ given by $f_-(u) = (u_1, \dots, u_n, -\sqrt{1 - \|u\|^2})$. Clearly, the intersection $S^n_+ \cap S^n_- = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$ is nonempty. Hence we conclude that S^n is connected

Some Examples

1. Empty set is connected, in fact vacuously so as it lacks non-empty subsets. Consequently it is not disconnected.
2. $\mathbb{R}_{\text{usual}}$ is connected, as is $\mathbb{R}^n_{\text{usual}}$ for all $n \in \mathbb{N}$.
3. The unit circle in $\mathbb{R}^2$ is connected.
4. $\mathbb{R}^2 \setminus \{(0,0)\}$ with its usual subspace topology is connected. Note that $\mathbb{R} \setminus \{0\}$ with its usual subspace topology is disconnected.
5. $\mathbb{R}^2 \setminus \{\text{the x-axis}\}$ is connected.
6. If $A \subseteq \mathbb{R}^2$ is countable, then $\mathbb{R}^2 \setminus A$ is connected. In particular, $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is connected. Note that it is the set of points such that at least one coordinate is irrational.
7. More generally, let $n > 1$ and let A be a countable subset of $\mathbb{R}^n$. Then $\mathbb{R}^n \setminus A$ is connected.
8. The discrete space X with more than one point is disconnected.
9. The annulus $\{x \in \mathbb{R}^2 : 1 < \|x\| < 2\}$ is connected. [Hint: Continuous image of a connected set is connected.]

Exercises

- Show that the circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is connected.
- Show that the following subsets of $\mathbb{R}^2$ are not connected:
  (a) $\{(x, y) \in \mathbb{R}^2 : xy \neq 0\}$.
  (b) $\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}$.
  (c) $\{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Q} \text{ and } y \notin \mathbb{Q}\}$.
- Show that the set $GL(2, \mathbb{R})$ is not connected.
- The $O(n, \mathbb{R})$ of orthogonal matrices of order n is not connected.
- Show that the set $SO(2, \mathbb{R}) := \{A \in O(2, \mathbb{R}) : \det A = 1\}$ is connected. [Hint: Write down all elements of $SO(2, \mathbb{R})$ explicitly.]
- Let X, Y be metric spaces. Assume that X is connected and that $f : X \to Y$ is continuous. Show that the graph

$$\Gamma_f := \{(x, y) \in X \times Y : y = f(x), x \in X\}$$

  is a connected subset of $X \times Y$ (with respect to the product metric).
- Let A be a nonempty connected subset of $\mathbb{R}$. Assume that every point of A is rational. What can you conclude?

Exercises

- Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ which take irrational values only at rational points and not at irrational points.
- Let $f, g : [0, 1] \to \mathbb{R}$ be continuous functions. Assume that $f(x) \in [0, 1]$ for all x and $g(0) = 0$ and $g(1) = 1$. Show that $f(x) = g(x)$ for some $x \in [0, 1]$.
- Let A be the union of the following subsets of $\mathbb{R}^2$:

$$\begin{aligned} S &:= \{(x, y) : x^2 + y^2 = 1\} \\ L_1 &:= \{(x, y) : x \geq 1 \text{ and } y = 0\} \\ L_2 &:= \{(x, y) : x \leq -1 \text{ and } y = 0\} \\ L_3 &:= \{(x, y) : y \geq 1 \text{ and } x = 0\} \\ L_4 &:= \{(x, y) : x \leq -1 \text{ and } x = 0\}. \end{aligned}$$

  Show that A is connected subset of $\mathbb{R}^2$. (Draw a picture of A!)
  Can you generalize this exercise?
- Let X be a (metric) space. Let S and $L_i (i \in I)$ be connected subsets of X. Assume that $S \cap L_i \neq \emptyset$. Show that $S \cup (U_{i \in I} L_i)$ is a connected subset of X. (This is a generalization of the last exercise!)

Exercises

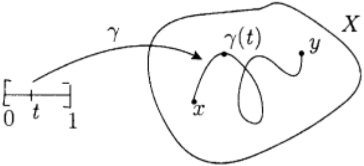
- We say that $f : X \to Y$ is a locally constant function if for each $x \in X$, there exists an open set U_x containing x with the property that f is a constant on U_x. If X is connected, then any locally constant function is constant on X.
- Let U be an open connected subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ be a differentiable function such that $Df(p) = 0$ for all $p \in U$. Then f is a constant function.
- Let $f : X \to \mathbb{R}$ be a nonconstant continuous function on a connected (metric) space. Show that $f(X)$ is uncountable and hence X is uncountable.
- Let (X, d) be a connected metric space. Assume that X has at least two elements. Then $|X| \geq |\mathbb{R}|$.
- Let (X, d) be an unbounded connected metric space. Let $x \in X$ and $r > 0$ be arbitrary. Show that there exists $y \in X$ such that $d(x, y) = r$.

Exercises

- Which of the following sets are connected subsets of $\mathbb{R}^2$?
  (a) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.
  (b) $\{(x, y) \in \mathbb{R}^2 : y = x^2\}$.
  (c) $\{(x, y) \in \mathbb{R}^2 : xy = 1\}$.
  (d) $\{(x, y) \in \mathbb{R}^2 : xy = c$ for some fixed $c \in \mathbb{R}\}$.
  (e) $\{(x, y) \in \mathbb{R}^2 : (x^2/a^2) + (y^2/b^2) = 1\}$ for some $a > b > 0$.
- Show that a circle or a line or a parabola in $\mathbb{R}^2$ is not homeomorphic to a hyperbola.
- Show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$. [Hint: Observe that if $f : X \to Y$ is a homeomorphism and if $f(A) = B$ for a subset $A \subseteq X$, the the restriction of f to $X \backslash A$ is a homeomorphism of $X \backslash A$ to $Y \backslash B$.]
- Let X be the union of axes given by $xy = 0$ in $\mathbb{R}^2$. Is it homeomorphic to a line, a circle, a parabola or the rectangular hyperbola $xy = 1$?
- Let $A \subseteq X$. What does it mean to say that the characteristic function χ_A continuous?
- Give an example of a sequence (A_n) of connected subsets of $\mathbb{R}^2$ such that $A_{n+1} \subseteq A_n$ for $n \in \mathbb{N}$, but $\cap_n A_n$ is not connected.
- Show that no nonempty open subset of $\mathbb{R}$ is homeomorphic to an open subset of $\mathbb{R}^2$.

Definition 20.

Let X be a metric space. A path in X is a continuous map $\gamma : [0,1] \to X$. If $\gamma(0) = x$ and $\gamma(1) = y$, then γ is said to be a path joining the points x and y or simply a path from x to y. We say that x is path connected to y if there is a path γ such that $\gamma(0) = x$ and $\gamma(1) = y$.



Every path is a uniformly continuous function, and its image is connected and compact.

1. Any point $x \in X$ is path connected to itself by a constant path $\gamma(t) = t$, for all $t \in [0,1]$.
2. If x is path connected to y, then y is path connected to x. Define $\sigma(t) := \gamma(1-t)$ for $t \in [0,1]$ which connects y to x. (The path σ is called the reverse path of γ.)
3. **Joining two paths into a single path :** If x is path connected to y and y is path connected to z in X, then x is path connected to z. More precisely, let $\gamma_i : [0,1] \to X$, $i = 1,2$, be two paths such that $\gamma_1(1) = \gamma_2(0)$. Then there exists a path $\gamma_3 : [0,1] \to X$ such that $\gamma_3(0) = \gamma_1(0)$, $\gamma_3(1/2) = \gamma_1(1) = \gamma_2(0)$ and $\gamma_3(1) = \gamma_2(1)$. We can consdier

$$\gamma_3(t) := \begin{cases} \gamma_1(2t) & \text{if } t \in [0,1/2] \\ \gamma_2(2t-1) & \text{if } t \in [1/2,1]. \end{cases}$$

Path Connected

The idea that any two points in a metric space can be joined by an unbroken curve in the space is perhaps a more intuitive idea of connectedness than the one we have adopted – at least it might be if we were not aware of space-filling curves. It is, however, a stronger concept than connectedness.

Definition 21.

A metric space X is said to be path connected if for any pair of points x and y in X, there exists a path $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Theorem 22.

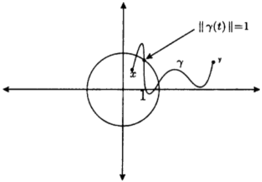
A metric space is path connected iff there exists a point $a \in X$ which is path connected to any $x \in X$.

Examples of Path Connected Metric Spaces

1. Any interval in $\mathbb{R}$ is path connected.
2. The space $\mathbb{R}^n$ is path connected. Any two points can be joined by a line segment: $\gamma(t) := x + t(y - x)$, for $0 \leq t \leq 1$. We call this path γ a **linear path**.
3. Any convex set in a normed linear space is path connected. Hence conclude that any open or closed ball in normed linear space is connected.
4. For every $r > 0$, the circle $C_r := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}$ is path connected.
5. The set $\{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } x^2 - y^2 = 1\}$ is path connected.
   The hyperbola $\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}$ is not path connected.
6. The parabola $\{(x, y) \in \mathbb{R}^2 : y^2 = x\}$ is path connected.
7. The union of the two parabolas $\{(x, y) \in \mathbb{R}^2 : y^2 = x\}$ and $\{(x, y) \in \mathbb{R}^2 : y = x^2\}$ is path connected.
8. The union of the two parabolas $\{(x, y) \in \mathbb{R}^2 : y^2 = x\}$ and $\{(x, y) \in \mathbb{R}^2 : y^2 = -x\}$ is path connected.
9. A non-empty intersection of connected subsets of a metric space need not be connected. The unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and the ellipse $\{z \in \mathbb{C} : 4(Re(z))^2 + (Im(z))^2 = 1\}$ are connected in $\mathbb{C}$. But their intersection is the two-point set $\{i, -i\}$ which is disconnected.

Path Connected

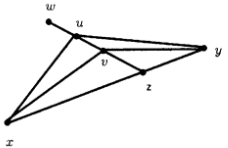
1. Let A and B be path connected subsets of a metric space with $A \cap B \neq \emptyset$. Then $A \cup B$ is path connected.
2. Any continuous image of a path connected space is path connected, that is, if $f : X \to Y$ is continuous and X is path connected, then $f(X)$ is path connected. In particular, path connectedness is a topological property.
3. Assume that a path $\gamma : [0, 1] \to \mathbb{R}^n$ connects a point $x \in B(0, 1) \subseteq \mathbb{R}^n$ to a point y with $\|y\| > 1$. Then there exists $t \in [0, 1]$ such that $\|\gamma(t)\| = 1$.



Proposition 23.

$\mathbb{R}^n \setminus \{0\}$ is path connected if $n \geq 2$.

Proof : Let $p = e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$ and $a = e_2 = (0, 1, 0, \ldots, 0) \in \mathbb{R}^n$. Let $x \in \mathbb{R}^n$ be any nonzero vector. Consider the line segments $[p, x]$ and $[x, q]$. We claim that at least one of them does not pass through the origin. If false, then $(1-t)p + tx = 0 = (1-s)q + sx$ for some $0 \leq s, t \leq 1$. From these equations, it follows that $(1-t)p = -tx$ and $(1-s)q = -sx$. Thus p and q are scalar maultiples of the same vector and hence they are linearly dependent. This contradiction shows that our claim is true. Note that by a similar reasoning, the line segment $[p, q]$ does not pass through the origin. Now consider the 'path' $[x, p]$ or the path $[x, q] \cup [q, p]$ connecting x and p, not passing through the origin. Thus any nonzero $x \in \mathbb{R}^n$ is path connected to p and hence $\mathbb{R}^n \setminus \{0\}$ is path connected.



Path Connected

1. The unit sphere $S^n := \{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$ is path connected. The sphere S^n is the continuous image of the path connected space $\mathbb{R}^{n+1} \setminus \{0\}$.
2. The annulus $\{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2\}$ is path connected. How about $\{x \in \mathbb{R}^2 : 1 < \|x\| < 2\}$?

Theorem 24.

Let X be a path connected metric space. Then X is connected.

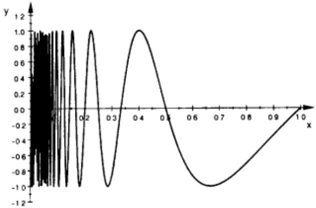
Proof. Let $f : X \to \{\pm\}$ be continuous. Fix $a \in X$. Let $x \in X$ be arbitrary. Since X is path connected, there exists a path $\gamma : [0,1] \to X$ such that $\gamma(0) = a$ and $\gamma(1) = x$. The function $f \circ \gamma : [0,1] \to \{\pm 1\}$ is continuous on the connected set $[0,1]$ and hence must be a constant. In particular, $f(a) = f \circ \gamma(0) = f \circ \gamma(1) = f(x)$. Since $x \in X$ is arbitrary, we have shown that f is a constant function. Therefore, X is connected.

Example 25 (Topologist's Sine Curve-I).

Consider

$$X := \{(x, \sin(\pi/x)) : 0 < x \leq 1\} \cup \{(0, y) : -1 \leq y \leq 1\} = A \cup B \text{ (say.)}$$

Then A is connected and $X = A \cup B = \overline{A}$, so X is connected. But X is not path connected because no point of B is path connected to any point of A.

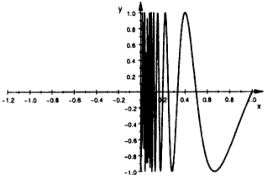


Example 26 (Topologist's Sine Curve-II).

Consider

$$X := \{(x, \sin(1/x)) : x > 0\} \cup \{(x, 0) : -1 \leq x \leq 0\} = A \cup B \text{ (say.)}$$

Clearly each of A and B is connected. Also, the point $(0, 0)$ is a limit point of the set A and hence $A_1 = A \cup \{(0, 0)\} \subseteq \overline{A}$ is connected. Since B and A_1 have a point in common, their union X is connected. But X is not path connected because there is no path connecting $(1/\pi, 0)$ and $(0, 0)$.



Path Connected

The following result is a typical application of connectedness argument and also provides a large class of path connected spaces.

Theorem 27.

Let U be an open connected subset of $\mathbb{R}^n$. Then U is path connected.

One could prove the result in a more general setting.

Theorem 28.

Let X be a connected metric space. Assume that each point of X has an open set U such that $x \in U$ and U is path connected. Then X is path connected.

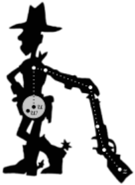
Exercises 29.

1. *Give at least two paths in $\mathbb{R}^2$ that connect $(-1, 0)$ and $(1, 0)$ and pass through $(0, 1)$.*
2. *Let A be a connected subset in $\mathbb{R}^n$ and $\varepsilon > 0$. Then the ε-neighbourhood of A defined by $U_\varepsilon(A) := \{x \in \mathbb{R}^n : d_A(x) < \varepsilon\}$ is path connected.*

Polygonally Connected

In a linear space, there are other forms of connectedness available to us. The simplest and strongest is convexity. But subsets of a linear space that are not convex may still be connected in a way that is, in general, stronger than pathwise connectedness. They may be polygonally connected.

Let S be a subset of a linear space S and $a, b \in S$. For any $n \in \mathbb{N}$, an n-tuple $(c_1, c_2, \ldots, c_n)$ of points of S is called a polygonal connection from a to b in S if $c_1 = a$ and $c_n = b$ and for each $i \in \{2, 3, \ldots, n\}$, the line segment $\{(1-t)c_{i-1} + tc_i : t \in [0, 1]\}$ is included in S.



Definition 30.

A vector space S is said to be polygonally connected if for each $a, b \in S$, there exists a polygonal connection from a to b in S.

- All convex subsets of a linear space are polygonally connected.
- The non-convex subset of $\mathbb{C}$ given by

$$S = \{z \in \mathbb{C} : |z - 1| \leq 1\} \cup \{z \in \mathbb{C} : |z + 1| \leq 1\}$$

  is polygonally connected because any two points that cannot be joined by a line segment in S can be joined by two such line segments meeting at the origin.
- The set $\{z \in \mathbb{C} : |z| = 1\}$ is pathwise connected but not polygonally connected.

Polygonally Connected

Theorem 31.

Every polygonally connected subset of a normed linear space X is path connected and therefore connected.

Theorem 32.

Let U be an open connected subset of a normed linear space X. Then U is polygonally connected and therefore path connected.

Example 33.

A closed connected subset of a normed linear space need not be even path connected. An example is the closure in $\mathbb{R}^2$ of the graph of the function $x \mapsto \sin(\frac{1}{x})$ defined on $(0, \infty)$.

Exercises

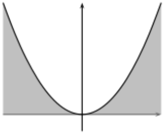
- Show that a ball of a connected metric space need not be connected.
- Suppose X is a metric space. Show that X is disconnected iff there is a non-emtpy proper subset S of X such that $\overline{S} \cap \overline{S^c} = \emptyset$.
- Give an example to show that the interior of a connected subset of a metric space need not be connected.
- Suppose X is a metric space and the number of connected components is finite. Show that each of them is both open and closed in X.
- Suppose S is proper closed subset of $[0,1]$ and $\{0,1\} \subseteq S$. Show that each connected component of $[0,1] \backslash S$ is an open interval (a,b) with $a, b \in S$.
- Is there any injective continuous funtion $\mathbb{R}^2$ to $\mathbb{R}$?

Exercises

- Show that the subset

$$S = \{(0,0)\} \cup \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 \leq x_1^2\}$$

of $\mathbb{R}^2$ is path connected but not polygonally connected.



$\{(0,0)\} \cup \{x \in \mathbb{R}^2 \mid 0 < x_2 \leq x_1^2\}$ is pathwise connected, but not polygonally connected

- Suppose X is a normed linear space and $\mathcal{C}$ is a chained collection of convex subsets of X. Show that

$$\bigcup_{C \in \mathcal{C}} C$$

is polygonally connected.

Locally Connected

We now consider topological spaces.

Recall that each topological space X is the set-theoretic disjoint union of its connected components, but in general (e.g. for $X = \mathbb{Q}$) fails to be the topological disjoint union. The problem is that the connected components in general are not open in X. We will seek to right this wrong here, by looking at a specific class of topological spaces.

Definition 34.

A topological space X is said to be locally connected *if*

- *for each open subset $U \subseteq X$ and $x \in U$, there exists a connected open subset V of X such that $x \in V \subseteq U$.*

Note that since U is open in X, any subset $V \subseteq U$ is open in U if and only if it is open in X. In such instances, we will sometimes abuse our language and say V is open without specifying the ambient space.

Locally Connected

One useful way to judge if a space is locally connected is as follows.

Proposition 35.

A topological space X is locally connected if and only if it has a basis all of whose elements are connected.

The key property we wish to prove is:

Theorem 36.

If X is locally connected, then every connected component of X is open in X. Hence X is the topological disjoint union of its connected components.

Locally Connected

Take our favourite topologist's sine curve $X = A \cup B$ where:

$$A = \{0\} \times [0,1] \quad \text{and} \quad B = \{(x, \sin(1/x)) : 0 < x \leq 1\}.$$

We saw that X is connected. However it is *not locally connected* since for any $0 < \varepsilon < 1$, the ε-neighbourhood of the origin contains more than one (in fact infinitely many!) connected components:

This also shows that **connected spaces are generally not locally connected**.

Corollary 37.

A locally connected space X is totally disconnected if and only if it is discrete.

Properties of Locally Connected Spaces

Similar to the case of locally compact, the following result explains why the property is local.

Proposition 38.

Let X be a topological space.

- *If $U \subseteq X$ is open, and X is locally connected, then so is U.*
- *If $X = \cup_i U_i$ is a union of open subsets and each U_i is locally connected, then so is X.*

It is not true that if $f : X \to Y$ is continuous and X is locally connected, then so is $f(X)$. Indeed, let $X = \mathbb{Q}$ with the discrete topology and $Y = \mathbb{Q}$ as a subspace of $\mathbb{R}$. The identity map on the underlying set $\mathbb{Q}$ then gives a surjective continuous map but Y is not locally connected. [Recall that if we replace locally connected with connected, then this is true.]

Properties of Locally Connected Spaces

Closed subsets of locally connected spaces are not locally connected in general. For example, take $X = \mathbb{R}$ and $Y = \{0\} \cup \{\frac{1}{n} : n = 1, 2, \ldots\}$. Then Y is closed in X, but there is no connected open subset of 0 in Y.

Proposition 39.

If X and Y are locally connected topological spaces, then so is $X \times Y$.

This does not hold for infinite products. For example, let $X = \{0, 1\}^N$, where $\{0, 1\}$ is given the discrete topology. We claim that X is totally disconnected.

Indeed, suppose some connected component Y contains $(x_n), (y_n) \in X$ where $x_n \neq y_n$ for some n. Then projecting to the n-th component gives a surjective map $\pi_n : Y \to \{0, 1\}$ which violates the theorem that the continuous image of a connected set is connected. Hence, X is a totally disconnected space which is not discrete (since it is compact by Tychonoff theorem), so it cannot be locally connected.

Examples of Locally Connected Spaces

1. Any discrete set is locally connected since we can take $V = \{x\}$.
2. Since the open intervals in $\mathbb{R}$ are connected, $\mathbb{R}$ has a basis of connected open subsets and is thus locally connected.
3. Hence $\mathbb{R}^n$ is also locally connected, as is any open subset of $\mathbb{R}^n$.
4. Let $X = [0, 1]$; then X is connected and locally connected. For example, at 0, any open subset must contain $[0, \varepsilon)$ for some $\varepsilon > 0$, which is open in X.
5. $\mathbb{Q}$ is totally disconnected yet not discrete, so it is not locally connected.

Locally Path Connected Spaces

Correspondingly, we have locally path connected spaces.

Definition 40.

A topological space X is locally path connected *if*

- *for each open subset $U \subseteq X$ and $x \in U$, there exists a path connected open subset V of X such that $x \in V \subseteq U$.*

The earlier properties all carry over since the proofs can be replicated by replacing path connected with connected whenever it appears in the notes.

Proposition 41.

A topological space X is locally path connected if and only if it has a basis all of whose elements are connected.

Locally Path Connected Spaces

Theorem 42.

If X is locally path connected, then every path connected component of X is open in X. Hence X is the topological disjoint union of its path connected components.

Proposition 43.

Let X be a topological space.

- *If $U \subseteq X$ is open, and X is locally path connected, then so is U.*
- *If $X = \cup_i U_i$ is a union of open subsets and each U_i is locally path connected, then so is X.*

Proposition 44.

If X and Y are locally path connected, then so is $X \times Y$.

Locally Path Connected Spaces

1. Any discrete set is locally path connected since we can take $V = \{x\}$.
2. Since the open intervals in $\mathbb{R}$ are path connected, $\mathbb{R}$ has a basis of path connected open subsets and is thus locally path connected.
3. Hence $\mathbb{R}^n$ is also locally path connected, as is any open subset of $\mathbb{R}^n$.
4. Let $X = [0, 1]$; then X is path connected and locally path connected. For example, at 0, any open subset must contain $[0, \varepsilon)$ for some $\varepsilon > 0$, which is open in X.
5. $\mathbb{Q}$ is totally disconnected; since each connected component is a disjoint union of path components, we see that the only path components of $\mathbb{Q}$ are $\{x\}$. Hence, $\mathbb{Q}$ is *not locally path connected*, for if it were, $\mathbb{Q}$ would have to be the disjoint union of all $\{x\}$ and hence discrete.

Let us see if we can find counter-examples similar to those for local connectedness.

1. **Not true: if X is path connected, then it is locally path connected**. Take the circle $S^1 = \{(x, y) \in \mathrm{R}^2 : x^2 + y^2 = 1\}$ and consider an infinite sequence of wheel spokes: $X_0 = [0, 1] \times \{0\}$ and: $X_n = \{(t \cos \frac{\pi}{n}, t \sin \frac{\pi}{n}) : \frac{n}{n+1} \leq t \leq 1\}$, for $n = 1, 2, \ldots$.
   Now take $X := S^1 \cup \left(U_{n=0}^{\infty} X_n\right)$:
   Now there is no path connected open subset V such that $(0, 0) \in V \subseteq N_X((0, 0), \frac{1}{2})$. On the other hand, X is clearly path connected.
2. **Not true: if $f : X \to Y$ is continuous and X is locally path connected, then so is Y.** Same example as before: let $X = \mathbb{Q}$ with the discrete topology and $Y = \mathbb{Q}$ as a subspace of $\mathbb{R}$. Then the identity map on $\mathbb{Q}$ is continuous and X is locally path connected, but $f(X) = Y$ is not.
3. **Not true: a product of infinitely many locally path connected spaces is locally path connected.**
   Same example as before: $X = \{0, 1\}^{\mathbb{N}}$, where $\{0, 1\}$ is given the discrete topology. We saw earlier that X is totally disconnected, so the components (and hence path components) are singleton sets. On the other hand, X is not discrete, so it cannot be locally path connected.

Relationship Between Locally Connected & Locally Path Connected

Finally, let us examine the relationship between the two notions. Since path connected sets are connected, we have:

1. **A locally path connected space is also locally connected.**
   The converse is not true.
2. **A locally connected space is not locally path connected in general.**
   This is hard: one can find a counter-example in *Munkres*, *Topology*, 2nd edition, page 162, chapter 25, exercise 3.
3. **If X is connected and locally path connected, then it is path connected.**
   Pick any path component Y of X. Since X is locally path connected, Y is open in X. The complement $X \setminus Y$ is a union of path components, each open in X, so it is open too. Thus Y is a clopen (both open and closed) subset of X and we must have $Y = X$.

References

- S. Kumaresan, *Topology of Metric Spaces*, Narosa Publishing House.
- Shirali, Satish, Vasudeva, Harkrishan L., *Metric Spaces*, Springer.
- Ó Searcóid, Mícheál, *Metric Spaces*, Springer.